

Weak collapsing and geometrisation of aspherical 3-manifolds

L. Bessières, G. Besson, M. Boileau, S. Maillot, J. Porti

February 1, 2008

Abstract

Let M be a closed, orientable, irreducible, non-simply connected 3-manifold. We prove that if M admits a sequence of Riemannian metrics whose sectional curvature is locally controlled and whose thick part becomes asymptotically hyperbolic and has a sufficiently small volume, then M is Seifert fibred or contains an incompressible torus. This result gives an alternative approach for the last step in Perelman's proof of the Geometrisation Conjecture for aspherical 3-manifolds.

Introduction

Thurston's Geometrisation Conjecture states that any closed, orientable, irreducible 3-dimensional manifold M is hyperbolic, Seifert fibred, or contains an incompressible torus. This conjecture has been proved recently by G. Perelman [31, 33, 32] (see also [27, 29, 8]) using R. Hamilton's Ricci flow. In this paper, we shall be concerned with the case where $\pi_1 M$ is nontrivial. Our results apply in particular if $\pi_1 M$ is infinite, which under the above hypotheses is equivalent to M being aspherical.

The last step of Perelman's proof in this case relies on a 'collapsing theorem' which is independent of the Ricci flow part. This result is stated without proof as Theorem 7.4 in [33]. A version of this theorem for closed 3-manifolds is given in the appendix of [36] using deep results of Alexandrov space theory, including Perelman's stability theorem (see [26]) and a fibration theorem for Alexandrov spaces [39].

Our main result, Theorem 0.1 below, implies Theorem 7.4 of [33] for closed, irreducible 3-manifolds which are not simply connected, and is sufficient to apply Perelman's construction of Ricci flow with surgery to geometrize these manifolds. The proof of Theorem 0.1

combines arguments from Riemannian geometry, algebraic topology, and 3-manifold theory. It uses Thurston's hyperbolisation theorem for Haken manifolds, but avoids the stability and fibration theorems for Alexandrov spaces.

In this text we call *hyperbolic manifold* a complete 3-manifold with constant sectional curvature equal to -1 and *finite volume*. The hyperbolic metric, which is unique up to isometry by Mostow rigidity, will be denoted by g_{hyp} .

In the next two definitions, M is a 3-manifold.

Definition. Let g be a Riemannian metric on M and $\varepsilon > 0$ be a real number. A point $x \in M$ is ε -thin with respect to g if there exists $0 < \rho \leq 1$ such that on the ball $B(x, \rho)$, the sectional curvature is greater than or equal to $-\rho^{-2}$ and the volume of this ball is less than $\varepsilon \rho^3$. Otherwise we say that x is ε -thick with respect to g . The set of ε -thin points (resp. ε -thick points) is called the ε -thin part (resp. ε -thick part) of M .

The following is a technical condition which guarantees the regularity of certain limits of riemannian manifolds.

Definition. Let g_n be a sequence of Riemannian metrics on M . We say that g_n has *locally controlled curvature in the sense of Perelman* if it has the following property: for all $\varepsilon > 0$ there exist $\bar{r}(\varepsilon) > 0, K_0(\varepsilon), K_1(\varepsilon) > 0$, such that for n large enough, if $0 < r \leq \bar{r}(\varepsilon)$, $x \in (M, g_n)$ satisfies $\text{vol}(B(x, r))/r^3 \geq \varepsilon$, and the sectional curvature on $B(x, r)$ is greater than or equal to $-r^{-2}$, then $|\text{Rm}(x)| < K_0 r^{-2}$ and $|\nabla \text{Rm}(x)| < K_1 r^{-3}$.

Next we define a topological invariant $V_0(M)$ which is essential to this paper. Let M be a closed 3-manifold. For us, a *link* in M is a (possibly empty, possibly disconnected) closed 1-submanifold of M . A link is *hyperbolic* if its complement is a hyperbolic 3-manifold. The invariant $V_0(M)$ is defined as the infimum of the volumes of all hyperbolic links in M . This quantity is finite because any closed 3-manifold contains a hyperbolic link [30]. Since the set of volumes of hyperbolic 3-manifolds is well-ordered, this infimum is always realised by some hyperbolic 3-manifold H_0 ; in particular, it is positive. We note that M is hyperbolic if and only if $H_0 = M$; indeed, every hyperbolic Dehn filling on a hyperbolic manifold strictly decreases the volume [4].

We now state the main result of this article:

Theorem 0.1. *Let M be a non-simply connected, closed, orientable, irreducible 3-manifold. Suppose that there exists a sequence g_n of Riemannian metrics on M satisfying:*

- (1) *The sequence $\text{vol}(g_n)$ is bounded.*
- (2) *Let $\varepsilon > 0$ be a real number and $x_n \in M$ be a sequence such that for all n , x_n is ε -thick with respect to g_n . Then the sequence of pointed manifolds (M, g_n, x_n) has a subsequence that converges in the C^2 topology towards a hyperbolic pointed manifold with volume strictly less than $V_0(M)$.*
- (3) *The sequence g_n has locally controlled curvature in the sense of Perelman.*

Then M is a graph manifold or contains an incompressible torus.

Recall that if M is a graph manifold, then M is Seifert fibred or contains an incompressible torus. Hence the conclusion of Theorem 0.1 implies that M satisfies the conclusion of the Geometrisation Conjecture as stated at the beginning of this paper.

Note that Hypothesis (2) of Theorem 0.1 may be vacuous; this is in particular the case if there is a sequence $\varepsilon_n \rightarrow 0$ such that for all n , every point of M is ε_n -thin with respect to g_n . In this situation, we shall say that the sequence g_n *collapses*. Thus Theorem 0.1 can be viewed as a *weak collapsing* result in the sense that we allow the thick part to be nonempty, but require a control on its volume. In the special case where g_n collapses, the proof of Theorem 0.1 does not use Hypothesis (1), and leads to the conclusion that M is a graph manifold. Hence we also obtain the following version of Perelman's collapsing theorem:

Corollary 0.2. *Let M be a closed, orientable, irreducible, non-simply connected 3-manifold. If M admits a sequence of Riemannian metrics that collapses and has controlled curvature in the sense of Perelman, then M is a graph manifold.*

Sequences of metrics satisfying the hypotheses of Theorem 0.1 are provided by Perelman's construction and study of the Ricci flow with surgery [33, 32]. From these deep results and Theorem 0.1 we deduce characterisations of hyperbolic and graph 3-manifolds: these are Theorems 0.3 and 0.4. For this we shall use the invariant $\overline{R}(M)$ defined below, which was first suggested by M. Anderson [3] (cf. also [33, §8] and [27, §93].)

Let M be a closed 3-manifold. If g is a Riemannian metric on M , we denote by $R_{\min}(g)$ the minimum of the scalar curvature R of g on M , and by $\hat{R}(g)$ the scale invariant quantity $R_{\min}(g) \text{vol}(g)^{2/3}$.

Note that if M has a hyperbolic metric g_{hyp} , then $\hat{R}(g_{hyp})$ is equal to $-6 \operatorname{vol}(g_{hyp})^{2/3}$.

We let $\overline{R}(M)$ be the (possibly infinite) supremum of $\hat{R}(g)$, taken over all Riemannian metrics g on M .¹

Theorem 0.3. *Let M be a closed, orientable, irreducible 3-manifold.*

- (1) *If $\overline{R}(M) \leq -6V_0(M)^{2/3}$, then M is hyperbolic and $\overline{R}(M) = \hat{R}(g_{hyp})$.*
- (2) *If $-6V_0(M)^{2/3} < \overline{R}(M)$, then M contains an incompressible torus or is Seifert fibred.*

Theorem 0.3 immediately implies the Geometrization Conjecture. Moreover, it shows that if M is a closed hyperbolic 3-manifold, then the hyperbolic metric realises the maximum in $\overline{R}(M)$, hence has the smallest volume among all complete metrics with scalar curvature bounded below by -6 . See [1] for an application.

Assertion (1) of Theorem 0.3 follows from the proof of Theorem 2.9 of [2]. We shall give another proof based on Thurston's hyperbolic Dehn filling theorem, following [35, Chap. 3]. When M is aspherical, Assertion (2) is proved using Theorem 0.1 and results of Perelman on the long time behaviour of Ricci flow with surgery [33]. (In fact, it is enough to assume that M is not simply connected.) For completeness, we have included in the statement the case where $\pi_1 M$ is finite, which follows from [32, 12, 29].

Our last result is a complement to Theorem 0.3. Let $V'_0(M)$ be the minimum of the volumes of all hyperbolic submanifolds $H \subset M$ having the property that H is a link complement or $\partial \bar{H}$ has at least one incompressible component.

Theorem 0.4. *Let M be a closed, orientable, irreducible 3-manifold. Then M is a graph manifold if and only if $\overline{R}(M) > -6V'_0(M)^{2/3}$.*

This article is organized as follows: in Section 1, we describe the strategy of the proof of Theorem 0.1. This proof is then given in Sections 2–4. In Section 5, we recall the necessary results coming from the Ricci flow. In Section 6, we prove Theorems 0.3 and 0.4.

Acknowledgments The authors wish to thank John Morgan for pointing out the proofs of Proposition 4.2 and Corollary 4.3 which allowed to extend Theorem 0.1 from the aspherical case to the non-simply connected case.

We also wish to thank the Clay Mathematics Institute for its financial support, as well as the project FEDER-MEC MTM2006-04353.

¹When M is aspherical, it turns out that $\overline{R}(M)$ is finite, equal to the Yamabe invariant of M (see e.g. [27, Section 93]), but we will not use this fact.

1 Sketch of proof of Theorem 0.1

For classical 3-manifold theory, we use [25], [23] as main references, as well as [6] for post-Thurston results. To avoid any confusion between metric balls and topological balls, we shall call *3-ball* a 3-manifold homeomorphic to the closed unit ball in \mathbf{R}^3 . By contrast, our metric balls $B(x, r)$ are open.

Throughout the paper we work in the smooth category. Recall that a *Haken manifold* is a connected, compact, orientable, irreducible 3-manifold which contains an incompressible surface. Any connected, compact, orientable, irreducible 3-manifold whose boundary is not empty is Haken. It follows from deep work of W. Thurston and earlier work of Jaco-Shalen and Johannson that every Haken manifold has a canonical decomposition along incompressible tori into Seifert and hyperbolic pieces (see e.g. the references given in [6].) We call this the *geometric decomposition* of the Haken manifold M . Moreover, a Haken manifold is a graph manifold if and only if all pieces in its geometric decomposition are Seifert.

Another key notion used in the proof of Theorem 0.1 is the *simplicial volume*, sometimes called Gromov's norm, introduced by M. Gromov in [17]. Our proof relies on an additivity result for the simplicial volume under gluing along tori (see [17, 28, 37]) which implies that the simplicial volume of a 3-manifold admitting a geometric decomposition is proportional to the sum of the volumes of the hyperbolic pieces. In particular, such a manifold has zero simplicial volume if and only if it is a graph manifold.

We also use in an essential way Gromov's vanishing theorem [17, 24]: if a n -dimensional closed manifold M can be covered by open sets U_i such that the covering has dimension less than n and the image of the canonical homomorphism $\pi_1 U_i \rightarrow \pi_1 M$ is amenable for all i , then the simplicial volume of M vanishes. (Recall that the dimension of a finite covering U_i is the dimension of its nerve.)

Below we outline the proof of Theorem 0.1. For simplicity, we explain it in the special case where the sequence g_n collapses. In the last paragraph, we shall say a few words about the general case.

Before discussing the proof proper, we give an example of a *covering argument* which can be used to deduce topological information on M (namely that M has zero simplicial volume) from the collapsing hypothesis.

For n large enough, thanks to the local control on the curvature, each point has a neighbourhood in (M, g_n) which is close to a metric ball in some manifold of nonnegative sectional curvature, and whose volume is small compared to the cube of the radius. These neigh-

neighbourhoods will be called *local models*. From the classification of manifolds with nonnegative sectional curvature, we deduce that these local models have virtually abelian, hence amenable fundamental groups. A technique introduced by Gromov [17] yields a covering of M , whose dimension is at most 2, by open sets contained in these neighbourhoods. As a consequence, Gromov's vanishing theorem implies that the simplicial volume of M vanishes.

The previous scheme, together with the additivity of the simplicial volume under gluing along incompressible tori, shows that a manifold which admits a geometric decomposition and a sequence of collapsing metrics is a graph manifold. This is however insufficient to prove Theorem 0.1 since we do not assume that M admits a geometric decomposition! Hence we need a trick similar to those of [7] and [5], which we now explain.

In the first step, we find a local model U such that all connected components of $M \setminus U$ are Haken. This requirement is equivalent to irreducibility of each component of $M \setminus U$. Since M is irreducible, it suffices to show that U is not contained in a 3-ball. This is in particular the case if U is *homotopically nontrivial*, i.e. the homomorphism $\pi_1(U) \rightarrow \pi_1(M)$ has nontrivial image.

The proof of the existence of a homotopically nontrivial local model U is done by contradiction: assuming that all local models are homotopically trivial, we construct a covering of M of dimension less than or equal to 2 by homotopically trivial open sets. By a result of J.C. Gómez-Larrañaga and F. González-Acuña [14], a closed irreducible 3-manifold admitting such a covering must have trivial fundamental group. This is where we use the hypothesis that M is not simply connected.

The second step, which is again a covering argument but done relatively to some fixed homotopically nontrivial local model U , shows that any manifold obtained by Dehn filling on $Y := M \setminus U$ has a covering of dimension less than or equal to 2 by virtually abelian open sets, and therefore has vanishing simplicial volume. We conclude using Proposition 4.14, which states that if Y a Haken manifold with boundary a collection of tori and such that the simplicial volume of every Dehn filling on Y vanishes, then Y is a graph manifold.² This finishes the sketch of proof of Theorem 0.1 in the collapsing case.

In this text, we shall not separate the case when g_n collapses, but we shall treat directly the general case. This implies that we first need to cover the thick part by submanifolds H_n^i approximating compact

²As already mentioned in [7, 5], Proposition 4.14 is a consequence of the geometrisation of Haken manifolds, additivity of the simplicial volume mentioned above, and Thurston's hyperbolic Dehn filling theorem.

cores of limiting hyperbolic manifolds (Section 2). We then cover the thin part by local models (Section 3). The bulk of the proof is in Section 4: assuming that M contains no incompressible tori, we consider the covering of M by approximately hyperbolic submanifolds and local models of the thin part and perform two covering arguments: the first one shows that at least one of these open subsets is homotopically nontrivial in M ; the second one is done relatively to this homotopically nontrivial subset and proves that M is a graph manifold.

2 Structure of the thick part

Until Section 4, we consider a 3-manifold M and a sequence of Riemannian metrics g_n satisfying the hypotheses of Theorem 0.1. For the sake of simplicity, in the sequel we use the notation $M_n := (M, g_n)$. The goal of this section is to describe the thick part of the manifolds M_n and to make the link between the topology of the thick part and the topology of M . We denote by $M_n^-(\varepsilon)$ the ε -thin part of M_n , and by $M_n^+(\varepsilon)$ its ε -thick part.

2.1 Covering the thick part

Proposition 2.1. *Up to taking a subsequence of M_n , there exists a finite (possibly empty) collection of pointed hyperbolic manifolds $(H^1, *^1), \dots, (H^m, *^m)$ and for every $1 \leq i \leq m$ a sequence $x_n^i \in M_n$ satisfying:*

- (i) $\lim_{n \rightarrow \infty} (M_n, x_n^i) = (H^i, *^i)$ in the \mathcal{C}^2 topology.
- (ii) For all sufficiently small $\varepsilon > 0$, there exist $n_0(\varepsilon)$ and $C(\varepsilon)$ such that for all $n \geq n_0(\varepsilon)$ one has $M_n^+(\varepsilon) \subset \bigcup_i B(x_n^i, C(\varepsilon))$.

Proof. By assumption, the sequence $\text{vol}(M_n)$ is bounded above. Let $\mu_0 > 0$ be a universal number such that any hyperbolic manifold has volume at least μ_0 .

If for all $\varepsilon > 0$ we have $M_n^+(\varepsilon) = \emptyset$ for n large enough, then Proposition 2.1 is vacuously true. Otherwise, we use Hypothesis (2) of Theorem 0.1: up to taking a subsequence of M_n , there exists $\varepsilon_1 > 0$ and a sequence of points $x_n^1 \in M_n^+(\varepsilon_1)$ such that (M_n, x_n^1) converges to a pointed hyperbolic manifold $(H^1, *^1)$.

If for all $\varepsilon > 0$ there exists $C(\varepsilon)$ such that, for n large enough, $M_n^+(\varepsilon)$ is included in $B(x_n^1, C(\varepsilon))$, then we are done. Otherwise there exists $\varepsilon_2 > 0$ and a sequence $x_n^2 \in M_n^+(\varepsilon_2)$ such that $d(x_n^1, x_n^2) \rightarrow \infty$. Again Hypothesis (2) of Theorem 0.1 ensures that, after taking a

subsequence, the sequence (M_n, x_n^2) converges to a pointed hyperbolic manifold $(H^2, *^2)$.

Note that for each i , and for n sufficiently large, M_n contains a submanifold \mathcal{C}^2 -close to some compact core of H_i and whose volume is greater than or equal to $\mu_0/2$. Moreover, for n fixed and large, these submanifolds are pairwise disjoint. Since the volume of the manifolds M^n is uniformly bounded above this construction has to stop. Condition (ii) of the conclusion of Proposition 2.1 is then satisfied for $0 < \varepsilon < \varepsilon_k$. \square

Remark. By Proposition 2.1 one can choose sequences $\varepsilon_n \rightarrow 0$ and $r_n \rightarrow \infty$ such that the ball $B(x_n^i, r_n)$ is arbitrarily close to a metric ball $B(*^i, r_n) \subset H^i$, for $i = 1, \dots, m$, and every point of $M_n \setminus \bigcup_i B(x_n^i, r_n)$ is ε_n -thin.

Let us fix a sequence of positive real numbers $\varepsilon_n \rightarrow 0$. Let H^1, \dots, H^m be hyperbolic limits given by Proposition 2.1. For each i we choose a compact core \bar{H}^i for H^i and for each n a submanifold \bar{H}_n^i and an approximation $\phi_n^i : \bar{H}_n^i \rightarrow \bar{H}^i$. Up to renumbering, one can assume that for all n we have $M_n \setminus \bigcup \bar{H}_n^i \subset M_n^-(\varepsilon_n)$, and that the \bar{H}_n^i 's are disjoint.

The hypothesis that the volume of each hyperbolic limit H^i is less than V_0 implies that for n sufficiently large no component \bar{H}_n^i is homeomorphic to the exterior of a link in M .

The logic of the proof is the following: each boundary component of \bar{H}_n^i is a torus. If one of those tori is incompressible, then the conclusion of Theorem 0.1 is true. The interesting case is when all the tori that appear in the boundary of the thick part are compressible. The remainder of this section is devoted to the two following results:

Proposition 2.2. *Up to taking a subsequence, one of the following properties is satisfied:*

- (i) *There exists an integer $i_0 \in \{1, \dots, m\}$ such that $\partial \bar{H}_n^{i_0}$ contains an incompressible torus for all n , or*
- (ii) *for all $i \in \{1, \dots, m\}$, \bar{H}_n^i is embedded in a solid torus or in a 3-ball contained in M_n for all n .*

Proposition 2.3. *If Conclusion (ii) of Proposition 2.2 is satisfied, then either M is a lens space or there exists for each n a submanifold $W_n \subset M_n$ such that:*

- (i) $\bigcup \bar{H}_n^i \subset W_n$.
- (ii) *Each connected component of W_n is a solid torus, or contained in a 3-ball and homeomorphic to the exterior of a knot in S^3 .*

(iii) The boundary of each component of W_n is a component of $\bigcup_i \partial \bar{H}_n^i$.

Subsection 2.2 is devoted to general topological results concerning compressible tori in 3-manifolds and abelian submanifolds. Proposition 2.2 and 2.3 will be proved in the Subsection 2.3.

2.2 Submanifolds with compressible boundary

Let X be an orientable, irreducible 3-manifold and T be a compressible torus embedded in X . The Loop Theorem shows the existence of a *compression disc* D for T , that is, a disc D embedded in M such that $D \cap T = \partial D$ and the curve ∂D is not null homotopic in T . By cutting open T along an open small regular neighbourhood of D and gluing two parallel copies of D along the boundary curves, one constructs an embedded 2-sphere S in X . We say that S is obtained by *compressing* T along D .

Since X is assumed to be irreducible, S bounds a 3-ball B . There are two possible situations depending on whether B contains T or not. The following lemma collects some standard results that we shall need.

Lemma 2.4. *Let X be an orientable, irreducible 3-manifold and T be a compressible torus embedded in X . Let D be a compression disc for T , S be a sphere obtained by compressing T along D , and B a ball bounded by S . Then:*

- i) $X \setminus T$ has two connected components U, V , and D is contained in the closure of one of them, say U .
- ii) If B does not contain T , then B is contained in \bar{U} , and \bar{U} is a solid torus.
- iii) If B contains T , then B contains V , and \bar{V} is homeomorphic to the exterior of a knot in S^3 . In this case, there exists a homeomorphism f from the boundary of $S^1 \times D^2$ into T such that the manifold obtained by gluing $S^1 \times D^2$ to \bar{U} along f is homeomorphic to X .

Remark. If T is a component of ∂X and T is a compressible torus, the same argument shows that X is a solid torus.

Lemma 2.5. *Let X be a closed, orientable, irreducible 3-manifold. Let $\bar{H} \subset X$ be a connected, compact, orientable, irreducible submanifold of X whose boundary is a collection of compressible tori. If \bar{X} is not homeomorphic to the exterior of a (possibly empty) link in X , then \bar{H} is included in a connected submanifold Y whose boundary is one of the tori of $\partial \bar{H}$ and which satisfies one of the following properties:*

- (i) Y is a solid torus, or
- (ii) Y is homeomorphic to the exterior of a knot in S^3 and contained in a ball $B \subset X$.

Proof. By hypothesis the boundary of \bar{H} is not empty. We denote by T_1, \dots, T_m the components of $\partial\bar{H}$. If one of them bounds a solid torus containing \bar{H} , we can choose this solid torus as Y . Henceforth we assume that this is not the case.

Each T_j being compressible, it separates and thus bounds a submanifold V_j not containing \bar{H} . Up to renumbering the boundary components of \bar{H} , we may assume that V_1, \dots, V_k are solid tori, but not V_{k+1}, \dots, V_m . At least one of the V_j 's is not a solid torus, otherwise \bar{H} would be homeomorphic to the exterior of a link in X .

For the same reason, at least one V_j , for some $j > k$, is not contained in a 3-ball. Otherwise each of the V_{k+1}, \dots, V_m is homeomorphic to the exterior of a knot in S^3 , by Lemma 2.4, and one could then replace each V_j , $k+1 \leq j \leq m$ by a solid torus without changing the topological type of X . Hence \bar{H} would be homeomorphic to the exterior of a link in X .

Pick a V_j , for $j > k$, which is not contained in a ball. Then compressing surgery on the torus $T_j = \partial V_j$ yields a sphere S bounding a ball B in X , which contains \bar{H} by the choice of V_j . This shows that conclusion (ii) is satisfied with $Y = X \setminus \text{int}V_j$. \square

2.3 Proof of Propositions 2.2 and 2.3

Proof of 2.2. If Assertion (i) of Proposition 2.2 is not satisfied, then, up to a subsequence, one may assume that for all $i \in 1, \dots, m$ and for all n , each component of $\partial\bar{H}_n^i$ is compressible in M . We fix an integer $i \in 1, \dots, m$. From the hypotheses of Theorem 0.1, we get the inequality $\text{vol}(H^i) < V_0(M)$, which implies that H^i is not homeomorphic to the complement of a link in M . In particular since \bar{H}_n^i is homeomorphic to the compact core of H^i , it is not homeomorphic to the exterior of a link in M . Lemma 2.5 allows to conclude that Assertion (ii) of Proposition 2.2 holds true. \square

Proof of 2.3. For each n and each $i \in 1, \dots, m$, we choose a submanifold Y_n^i containing \bar{H}_n^i , given by Lemma 2.5. We take W_n to be the union of the Y_n^i . Then Assertion (i) of Proposition 2.3 is straightforward.

Assume that M is not a lens space. Then M_n cannot be the union of two submanifolds Y_n^i and Y_n^j , otherwise M_n can be covered either by two solid tori, or by a solid torus and a ball or by two balls. In the first case M_n would be homeomorphic to a lens space by [16], while

in the other two cases M_n would be covered by three balls and thus homeomorphic to the 3-sphere S^3 by [22], see also [15]. Thus for all i_1, i_2 , the submanifolds $Y_n^{i_1} Y_n^{i_2}$ are disjoint or one contains the other, because they have disjoint boundaries. In this case each component of W_n is homeomorphic to one of the Y_n^i . This yields Assertions (ii) and (iii) of Proposition 2.3. \square

3 Local structure of the thin part

In this section, it is implicit that any quantity depending on a point $x \in M_n$ is computed with respect to the metric g_n on M_n and thus depends also on n .

Let us choose a sequence $\varepsilon_n \rightarrow 0$ (see the remark after Proposition 2.1). For all $x \in M_n^-(\varepsilon_n)$, we choose a radius $0 < \rho(x) \leq 1$, such that on the ball $B(x, \rho(x))$ the curvature is $\geq -\rho^{-2}(x)$ and the volume of this ball is $< \varepsilon_n \rho^3(x)$.

In the following proposition we use Cheeger-Gromoll's soul theorem [10].

Proposition 3.1. *For all $D > 1$ there exists $n_0(D)$ such that if $n > n_0(D)$, then for all $x \in M_n^-(\varepsilon_n)$ we have the following alternative:*

- (a) *Either M_n is $\frac{1}{D}$ -close to some closed nonnegatively curved 3-manifold, or*
- (b) *there exists a radius $\nu(x) \in (0, \rho(x))$ and a complete noncompact Riemannian 3-manifold X_x , with nonnegative sectional curvature and soul S_x , such that the following properties are satisfied:*
 - (1) *$B(x, \nu(x))$ is $\frac{1}{D}$ -close to a metric ball in X_x .*
 - (2) *There exists an approximation $f_x : B(x, \nu(x)) \rightarrow X_x$ such that*

$$\max\{d(f(x), S_x), \text{diam } S_x\} \leq \frac{\nu(x)}{D}.$$

- (3) *$\text{vol}(B(x, \nu(x))) \leq \frac{1}{D} \nu^3(x)$.*

Remark. Since $\nu(x) < \rho(x)$, the sectional curvature on $B(x, \nu(x))$ is greater than or equal to $-\frac{1}{\rho^2(x)}$, which is in turn bounded below by $-\frac{1}{\nu^2(x)}$.

Remark. The only closed, orientable and irreducible 3-manifold containing a projective plane is RP^3 , which is a graph manifold. Therefore if the manifold M is not homeomorphic to RP^3 , then the soul S_x can be homeomorphic to a point, a circle, a 2-sphere, a 2-torus or a Klein bottle. In this case, the ball $B(x, \nu(x))$ is homeomorphic to B^3 , $S^1 \times D^2$, $S^2 \times I$, $T^2 \times I$ or to the twisted I -bundle on the Klein bottle.

Before starting the proof of this proposition, we prove the following lemma and its consequence:

Lemma 3.2. *There exists a universal constant $C > 0$ such that for all $\varepsilon > 0$, for all $x \in M_n$, and for all $r > 0$, if the ball $B(x, r)$ has volume $\geq \varepsilon r^3$ and curvature $\geq -r^{-2}$, then for all $y \in B(x, \frac{1}{3}r)$ and all $0 < r' < \frac{2}{3}r$, the ball $B(y, r')$ has volume $\geq C \cdot \varepsilon (r')^3$ and curvature $\geq -(r')^{-2}$.*

We use the function $v_{-\kappa^2}(r)$ to denote the volume of the ball of radius r in the 3-dimensional hyperbolic space with curvature $-\kappa^2$. Notice that $v_{-\kappa^2}(r) = \kappa^{-3}v_{-1}(\kappa r)$.

Proof. The lower bound on the curvature is a consequence of the monotonicity of the function $-r^{-2}$ with respect to r . In order to estimate from below the normalised volume we apply Bishop-Gromov's inequality twice. First to the ball around y , increasing the radius r' to $\frac{2}{3}r$:

$$\text{vol}(B(y, r')) \geq \text{vol}(B(y, \frac{2}{3}r)) \frac{v_{-r^{-2}}(r')}{v_{-r^{-2}}(\frac{2}{3}r)}.$$

Using that $v_{-r^{-2}}(r') = r^3 v_{-1}(\frac{r'}{r}) \geq r^3 \left(\frac{r'}{r}\right)^3 C_1$ for $C_1 > 0$ uniform, $v_{-r^{-2}}(\frac{2}{3}r) = r^3 v_{-1}(\frac{2}{3})$, and that the ball $B(y, \frac{2}{3}r)$ contains $B(x, \frac{1}{3}r)$, we have

$$\text{vol}(B(y, r')) \geq \text{vol}(B(x, \frac{1}{3}r)) \left(\frac{r'}{r}\right)^3 C_2.$$

Applying again the Bishop-Gromov inequality:

$$\text{vol}(B(x, \frac{1}{3}r)) \geq \text{vol}(B(x, r)) \frac{v_{-r^{-2}}(\frac{1}{3}r)}{v_{-r^{-2}}(r)} \geq r^3 \varepsilon \frac{v_{-1}(\frac{1}{3})}{v_{-1}(1)} = r^3 \varepsilon C_3.$$

Hence $\text{vol}(B(y, r')) \geq (r')^3 \varepsilon C_4$. \square

We deduce an ‘improvement’ of the controlled curvature in the sense of Perelman, in which the conclusion is valid at each point of some metric ball, not only the centre. The only price to pay is that the constants can be slightly different.

Corollary 3.3. *For all $\varepsilon > 0$ there exists $\bar{r}'(\varepsilon) > 0$, $K'_0(\varepsilon), K'_1(\varepsilon)$ such that for n large enough, if $0 < r \leq \bar{r}'(\varepsilon)$, $x \in M_n$ and the ball $B(x, r)$ has volume $\geq \varepsilon r^3$ and sectional curvatures $\geq -r^{-2}$ then, for all $y \in B(x, \frac{1}{3}r)$, $|\text{Rm}(y)| < K'_0 r^{-2}$ and $|\nabla \text{Rm}(y)| < K'_1 r^{-3}$.*

Proof. It suffices to apply Lemma 3.2, setting $\bar{r}'(\varepsilon) = \bar{r}(C\varepsilon)$, $K'_0(\varepsilon) = K_0(C\varepsilon)$ and $K'_1(\varepsilon) = K_1(C\varepsilon)$. \square

Proof of Proposition 3.1. Let us assume that there exists $D_0 > 1$ and, after re-indexing, a sequence $x_n \in M_n^-(\varepsilon_n)$ such that neither of the conclusions of Proposition 3.1 holds with $D = D_0$.

Set $\varepsilon_0 := \frac{1}{1000D_0}$. We shall rescale the metrics using the following radii:

Definition. For $x \in M_n$, define

$$\text{rad}(x) = \inf\{r > 0 \mid \text{vol}(B(x, r))/r^3 \leq \varepsilon_0\}.$$

We gather in the following lemma some properties which will be useful for the proof:

Lemma 3.4. (i) For n large enough and $x \in M_n^-(\varepsilon_n)$, one has $0 < \text{rad}(x) < \rho(x)$.

(ii) For n sufficiently large and $x \in M_n^-(\varepsilon_n)$, one has

$$\frac{\text{vol}(B(x, \text{rad}(x)))}{\text{rad}(x)^3} = \varepsilon_0.$$

(iii) For $L > 1$, there exists $n_0(L)$ such that for $n > n_0(L)$ and for $x \in M_n^-(\varepsilon_n)$ we have

$$L \text{rad}(x) \leq \rho(x).$$

$$\text{In particular } \lim_{n \rightarrow \infty} \text{rad}(x_n) = \lim_{n \rightarrow \infty} \frac{\text{rad}(x_n)}{\rho(x_n)} = 0.$$

Proof of Lemma 3.4. Property (i) follows from continuity, by comparing the limit $\text{vol}(B(x, \delta))/\delta^3 \rightarrow \frac{4}{3}\pi$ when $\delta \rightarrow 0$ with

$$\text{vol}(B(x, \rho(x)))/\rho(x)^3 < \varepsilon_n \rightarrow 0.$$

Assertion (ii) is also proved by continuity.

We prove (iii) for $L > 1$ using the function

$$f_x(s) = \frac{\text{vol}(B(x, s \text{rad}(x)))}{(s \text{rad}(x))^3}.$$

One has $f_x(1) = \varepsilon_0$; for all $s \in [1, L]$, $f_x(s) \geq \frac{\varepsilon_0}{s^3} \geq \frac{\varepsilon_0}{L^3}$. Furthermore, for $s < 1$, $f_x(s) > \varepsilon_0$ by the definition of $\text{rad}(x)$. It suffices then to choose n_0 so that for all $n \geq n_0$ one has $\varepsilon_n < \frac{\varepsilon_0}{L^3}$. \square

Remark. For n large enough, from the preceding Lemma, we have $\text{rad}(x_n) < \bar{r}'(\varepsilon_0)$.

Corollary 3.5. *There exists a constant $C > 0$ such that any sequence of $x_n \in M_n$ satisfies*

$$\frac{\text{inj}(x_n)}{\text{rad}(x_n)} \geq C$$

for n large enough.

Proof. Let us first remark that, since $\text{rad}(x_n) < \rho(x_n)$, the sectional curvatures on $B(x_n, \text{rad}(x_n))$ are $\geq -\frac{1}{\rho(x_n)^2} > -\frac{1}{\text{rad}(x_n)^2}$. Moreover, as $\text{rad}(x_n) < \bar{r}'(\varepsilon_0)$, Corollary 3.3 shows that the curvature on the ball $B(x_n, \frac{\text{rad}(x_n)}{3})$ is bounded above by $K'_0(\varepsilon_0)/\text{rad}(x_n)^2$. This rescaled ball

$$\frac{1}{\text{rad}(x_n)} B(x_n, \frac{1}{3} \text{rad}(x_n))$$

has radius ≤ 1 , volume $\geq \varepsilon_0 C_0$ (where C_0 is a universal constant coming from Bishop-Gromov) and curvatures $\leq K'_0(\varepsilon_0)$. Using Cheeger's propeller lemma [9, Thm. 5.8], the injectivity radius at the centre of the rescaled ball is bounded below by some constant $C > 0$. This proves Corollary 3.5. \square

Having proved Lemma 3.4 and its corollary, we continue the proof of Proposition 3.1. Let us consider the rescaled manifold $\overline{M}_n = \frac{1}{\text{rad}(x_n)} M_n$. We look for a limit of the sequence $(\overline{M}_n, \bar{x}_n)$, where \bar{x}_n is the image of x_n . The ball $B(\bar{x}_n, \frac{\rho(x_n)}{\text{rad}(x_n)}) \subset \overline{M}_n$ has sectional curvature bounded below by $-\left(\frac{\text{rad}(x_n)}{\rho(x_n)}\right)^2$, which goes to 0 when $n \rightarrow \infty$, as follows from Assertion (iii) of Lemma 3.4.

Given $L > 0$, the ball $B(\bar{x}_n, 3L)$ is obtained by rescaling the ball $B(x_n, 3L \text{rad}(x_n))$. Since $3L \text{rad}(x_n) < \rho(x_n)$, the sectional curvature on $B(x_n, 3L \text{rad}(x_n))$ is $\geq -\frac{1}{\rho(x_n)^2} \geq -\frac{1}{(3L \text{rad}(x_n))^2}$. Moreover, we have

$$\frac{\text{vol}(B(x_n, 3L \text{rad}(x_n)))}{(3L \text{rad}(x_n))^3} \geq \frac{\varepsilon_0}{(3L)^3}.$$

By applying Corollary 3.3 for n sufficiently large so that we have $3L \text{rad}(x_n) \leq \bar{r}'(\frac{\varepsilon_0}{(3L)^3})$, one gets that the curvature is locally controlled in the sense of Perelman at each point of the ball $B(x_n, L \text{rad}(x_n))$. Therefore the curvature and its first derivative can be bounded above on any ball $B(\bar{x}_n, L) \subset \overline{M}_n$ with a given radius $L > 0$.

Since the injectivity radius of the basepoint x_n is bounded below along the sequence, this upper bound on the curvature allows to use Gromov's compactness theorem [18, Chap. 8, Thm. 8.28], [34] and its versions with regularised limit [21, Thm. 2.3] or [13, Thm. 4.1 and 5.10]. It follows that the pointed sequence $(\overline{M}_n, \bar{x}_n)$ subconverges in the \mathcal{C}^2 -topology towards a 3-dimensional smooth manifold $(\overline{X}_\infty, x_\infty)$,

with a complete riemannian metric of class \mathcal{C}^2 with nonnegative sectional curvature. This limit manifold cannot be closed, because that would contradict the assumption that the conclusion of Proposition 3.1 does not hold.

Hence \overline{X}_∞ is not compact. Let \overline{S} be its soul. Let us choose

$$\nu(x_n) = L \operatorname{rad}(x_n) \quad \text{where } L \geq 2 \operatorname{diam}(\overline{S} \cup \{x_\infty\}) D_0.$$

for n large (to be specified later) we set

$$X_{x_n} = \operatorname{rad}(x_n) \overline{X}_\infty, \quad \text{and} \quad S_{x_n} = \operatorname{rad}(x_n) \overline{S}.$$

We then have

$$\operatorname{diam}(S_{x_n}) = \operatorname{rad}(x_n) \operatorname{diam}(\overline{S}) < \nu(x_n)/D_0.$$

Let $\bar{f}_n: B(\bar{x}_n, L) \rightarrow (\overline{X}_\infty, x_\infty)$ be a δ_n -approximation, where δ_n is a sequence going to 0. After rescaling $f_n: B(x_n, L \operatorname{rad}(x_n)) \rightarrow X_{x_n}$ is also a δ_n -approximation. We get:

$$\begin{aligned} d(f_n(x_n), S_{x_n}) &= \operatorname{rad}(x_n) d(\bar{f}_n(\bar{x}_n), \overline{S}) \\ &\leq \operatorname{rad}(x_n) (d(\bar{f}_n(\bar{x}_n), \bar{x}_\infty) + d(\bar{x}_\infty, \overline{S})) \\ &\leq \operatorname{rad}(x_n) \delta_n + \frac{\nu(x_n)}{2D_0} \leq \frac{\nu(x_n)}{D_0}. \end{aligned}$$

This proves assertion (2) of Proposition 3.1.

Using the fact that $\nu(x_n) = L \operatorname{rad}(x_n) < \rho(x_n)$, the curvature on $B(x_n, \nu(x_n))$ is $\geq -1/\nu(x_n)^2$, $L > 1$ and the Bishop-Gromov inequality, we get:

$$\begin{aligned} \frac{\operatorname{vol}(B(x_n, \nu(x_n)))}{v_{-\frac{1}{\nu^2(x_n)}}(\nu(x_n))} &\leq \frac{\operatorname{vol}(B(x_n, \operatorname{rad}(x_n)))}{v_{-\frac{1}{\nu^2(x_n)}}(\operatorname{rad}(x_n))} = \varepsilon_0 \frac{\operatorname{rad}(x_n)^3}{v_{-\frac{1}{\nu^2(x_n)}}(\operatorname{rad}(x_n))} = \\ &\varepsilon_0 \left(\frac{\operatorname{rad}(x_n)}{\nu(x_n)} \right)^3 \frac{1}{v_{-1}(\frac{\operatorname{rad}(x_n)}{\nu(x_n)})} = \varepsilon_0 \frac{1}{L^3} \frac{1}{v_{-1}(\frac{1}{L})}. \end{aligned}$$

Taking now L sufficiently large, we find that:

$$\operatorname{vol}(B(x_n, \nu(x_n))) \leq \varepsilon_0 \frac{1}{L^3} \frac{v_{-1}(1)}{v_{-1}(\frac{1}{L})} \nu^3(x_n) \leq 1000 \varepsilon_0 \nu^3(x_n) = \frac{1}{D_0} \nu^3(x_n),$$

where the last equality comes from the definition of ε_0 .

Hence we get the contradiction required to conclude the proof of Proposition 3.1. \square

4 Constructions of coverings

4.1 Embedding thick pieces in solid tori

We begin by making some reductions for the proof of Theorem 0.1.

If case (a) of Proposition 3.1 occurs, then M is a closed, orientable, irreducible 3-manifold admitting a metric of nonnegative sectional curvature. By [19, 20], M is spherical or Euclidean, hence a graph manifold. Therefore we may assume that all local models are noncompact.

For the same reasons, since lens spaces are graph manifolds, we can also assume that M is not homeomorphic to a lens space, and in particular does not contain a projective plane.

If there exists an integer $i_0 \in 1, \dots, m$ such that, up to a subsequence, $\partial \bar{H}_n^{i_0}$ contains an incompressible torus for all n , then Theorem 0.1 is proved. We thus assume that for all $i \in 1, \dots, m$ and for all n , each component of $\partial \bar{H}_n^i$ is compressible in M_n and thus Propositions 2.2 and 2.3 apply and give for each n a submanifold W_n .

Assume that there exists a component X of W_n which is not a solid torus. From Proposition 2.3(ii), X is a knot exterior and contained in a 3-ball $B \subset M_n$. By Lemma 2.4, it is possible to replace X by a solid torus Y without changing the global topology. Let us denote by M'_n the manifold thus obtained. We can endow M'_n with a Riemannian metric g'_n , equal to g_n away from Y and such that an arbitrarily large collar neighbourhood of ∂Y in Y is isometric to a collar neighbourhood ∂X in X . When n is large, this neighbourhood is thus almost isometric to a long piece of a hyperbolic cusp, and this geometric property will be sufficient for our covering arguments.

Repeating this construction for each component of W_n which is not a solid torus, we obtain a Riemannian manifold (M''_n, g''_n) together with a submanifold W''_n satisfying the following properties:

- (i) M''_n is homeomorphic to M_n .
- (ii) $M''_n \setminus W''_n$ is equal to $M_n \setminus W_n$ and the metrics g_n and g''_n coincide on this set.
- (iii) $M''_n \setminus W''_n = M_n \setminus W_n$ is ε_n -thin.
- (iv) When n goes to infinity, there exists a collar neighbourhood of $\partial W''_n$ in W''_n of arbitrarily large diameter isometric to the corresponding neighbourhood in W_n .
- (v) Each component of W''_n is a solid torus.

For simplicity, we use the notation M_n, g_n, W_n instead of M''_n, g''_n, W''_n . This amounts to assuming in the conclusion of Proposition 2.3 that all components of W_n are solid tori.

4.2 Existence of a homotopically nontrivial open set

We say that an arcwise connected set $U \subset M$ is *homotopically trivial* (in M) if the image of the homomorphism $\pi_1(U) \rightarrow \pi_1(M)$ is trivial. More generally, we say that the subset $U \subset M$ is homotopically trivial if all its arcwise connected components have this property.

We recall that the *dimension* of a finite covering $\{U_i\}_i$ of M is the dimension of its nerve, hence the dimension plus one equals the maximal number of U_i 's containing a given point.

Proposition 4.1. *There exists $D_0 > 0$ such that for all $D > D_0$, for every n greater than or equal to the number $n_0(D)$ given by Proposition 3.1, one of the following assertions is true:*

- (a) *some connected component of W_n is not homotopically trivial, or*
- (b) *there exists $x \in M_n \setminus \text{int}(W_n)$ such that the image of $\pi_1(B(x, \nu(x))) \rightarrow \pi_1(M_n)$ is not homotopically trivial.*

In [14] J.C. Gómez-Larrañaga and F. González-Acuña have computed the 1-dimensional Lusternik-Schnirelmann category of a closed 3-manifold. One step of their proof gives the following proposition (cf. [14, Proof of Prop. 2.1]:)

Proposition 4.2. *Let X be a closed, connected 3-manifold. If X has a covering of dimension 2 by open subsets which are homotopically trivial in X , then there is a connected 2-dimensional complex K and a continuous map $f : X \rightarrow K$ such that the induced homomorphism $f_* : \pi_1(X) \rightarrow \pi_1(K)$ is an isomorphism.* \square

Standard homological arguments show the following, cf. [14, §3]:

Corollary 4.3. *Let X be a closed, connected, orientable, irreducible 3-manifold. If X has a covering of dimension 2 by open subsets which are homotopically trivial in X , then X is simply connected.*

Proof. By Proposition 4.2, let $f : X \rightarrow K$ be a continuous map from X to a connected 2-dimensional complex K , such that the induced homomorphism $f_* : \pi_1(X) \rightarrow \pi_1(K)$ is an isomorphism. Let Z be a $K(\pi_1(X), 1)$ space. Let $\phi : X \rightarrow Z$ be a map from X to Z realizing the identity homomorphism on $\pi_1(X)$ and let $\psi : K \rightarrow Z$ be the map from K to Z realizing the isomorphism $f_*^{-1} : \pi_1(K) \rightarrow \pi_1(X)$. Then ϕ is homotopic to $\psi \circ f$ and the induced homomorphism $\phi_* : H_3(X; \mathbb{Z}) \rightarrow H_3(Z; \mathbb{Z})$ factors through $\psi_* : H_3(K; \mathbb{Z}) \rightarrow H_3(Z; \mathbb{Z})$. Since $H_3(K; \mathbb{Z}) = \{0\}$, the homomorphism ϕ_* must be trivial.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Z \\ f \downarrow & \nearrow \psi & \\ K & & \end{array}$$

If $\pi_1(X)$ is infinite, then X is aspherical and ϕ_* is an isomorphism. Therefore $\pi_1(X)$ is finite.

If $\pi_1(X)$ is finite of order $d > 1$, then let \tilde{X} be the universal covering of X . The covering map $p : \tilde{X} \rightarrow X$ induces an isomorphism between the homotopy groups $\pi_k(\tilde{X})$ and $\pi_k(X)$ for $k \geq 2$. Since $\pi_2(X) = \{0\}$, $\pi_2(\tilde{X}) = \{0\}$, and by the Hurewicz theorem, the canonical homomorphism $\pi_3(\tilde{X}) \rightarrow H_3(\tilde{X}; \mathbb{Z}) = \mathbb{Z}$ is an isomorphism. It follows that the canonical map $\pi_3(X) = \mathbb{Z} \rightarrow H_3(X; \mathbb{Z}) = \mathbb{Z}$ is the multiplication by the degree $d > 1$ of the covering $p : \tilde{X} \rightarrow X$. It is well known that one can construct a $K(\pi_1(X), 1)$ space Z by adding a 4-cell to kill the generator of $\pi_3(X) = \mathbb{Z}$, and adding further cells of dimension ≥ 5 to kill the higher homotopy groups. Then the inclusion $\phi : X \rightarrow Z$ induces the identity on $\pi_1(X)$ and a surjection $\phi_* : H_3(X; \mathbb{Z}) = \mathbb{Z} \rightarrow H_3(Z; \mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$. Therefore X must be simply connected. \square

In the proof of Proposition 4.1 we argue by contradiction using Corollary 4.3 and the fact that $\pi_1(M)$ is not trivial.

With the notation of Proposition 3.1, we may assume that for arbitrarily large D there exists $n \geq n_0(D)$ such that the image of $\pi_1(B(x, \nu(x))) \rightarrow \pi_1(M_n)$ is trivial for all $x \in M_n \setminus \text{int}(W_n)$ as well as for each component of W_n .

Then for all $x \in M_n \setminus \text{int}(W_n)$ we set:

$$\text{triv}(x) = \sup \left\{ r \left| \begin{array}{l} \pi_1(B(x, r)) \rightarrow \pi_1(M_n) \text{ is trivial and} \\ B(x, r) \text{ is contained in } B(x', r') \text{ with} \\ \text{curvature} \geq -\frac{1}{(r')^2} \text{ and } \frac{\text{vol}(B(x', r'))}{(r')^3} \leq 1/D \end{array} \right. \right\}$$

By hypothesis, we have $\text{triv}(x) \geq \nu(x)$. The proof of Proposition 4.1 follows by contradiction with the following assertion.

Assertion 4.4. *There exists a covering of M_n by open sets U_1, \dots, U_p such that:*

- *Each U_i is either contained in some $B(x_i, \text{triv}(x_i))$ or in a subset that deformation retracts to a component of $\text{int}(W_n)$. In particular, U_i is homotopically trivial in M .*
- *The dimension of this covering is at most 2.*

Since M is irreducible and non-simply connected, this contradicts Corollary 4.3.

To prove Assertion 4.4, we define

$$r(x) = \min\left\{\frac{1}{11} \text{triv}(x), 1\right\}.$$

Lemma 4.5. *Let $x, y \in M_n \setminus \text{int}(W_n)$. If $B(x, r(x)) \cap B(y, r(y)) \neq \emptyset$, then*

- (a) $3/4 \leq r(x)/r(y) \leq 4/3$;
- (b) $B(x, r(x)) \subset B(y, 4r(y))$.

Proof. We may assume that $r(x) \leq r(y)$ and that $r(x) = \frac{1}{11} \text{triv}(x) < 1$. From the triangle inequality, we get:

$$\text{triv}(x) \geq \text{triv}(y) - r(x) - r(y),$$

hence

$$11r(x) = \text{triv}(x) \geq 11r(y) - r(x) - r(y) \geq 9r(y).$$

Consequently, we have $1 \geq r(x)/r(y) \geq 9/11 \geq 3/4$, which shows (a).

Now (b) follows because $2r(x) + r(y) < 4r(y)$. \square

If n is sufficiently large, we can choose points $x_1, \dots, x_q \in \partial W_n$ in such a way that a tubular neighbourhood of each component of the boundary of W_n contains precisely one of the x_j 's and that the balls $B(x_j, 1)$ are disjoint, have volume $\leq \frac{1}{D}$ and sectional curvature close to -1 . Furthermore, we may assume that $B(x_j, 1)$ is included in a submanifold W'_n which contains W_n and can be retracted by deformation onto it. In particular W'_n and $B(x_j, 1)$ are homotopically trivial, since we have assumed that W_n is. This implies that $\text{triv}(x_j)$ is close to 1.

Moreover, for n large enough, we may assume that $B(x_j, \frac{2}{3}r(x_j))$ contains an almost horospherical torus corresponding to a boundary component of W_n . We can even arrange for both components of $B(x_j, r(x_j)) \setminus B(x_j, \frac{2}{3}r(x_j))$ to also contain a parallel almost horospherical torus, which allows to retract W_n on the complement of $B(x_j, \frac{2}{3}r(x_j))$.

We complete x_1, x_2, \dots, x_q to a sequence x_1, x_2, \dots in $M_n \setminus \text{int}(W_n)$ such that the balls $B(x_1, \frac{1}{4}r(x_1)), B(x_2, \frac{1}{4}r(x_2)), \dots$ are pairwise disjoint.

Such a sequence is necessarily finite, since $M_n \setminus \text{int}(W_n)$ is compact, and Lemma 4.5 implies a positive local lower bound for the function $x \mapsto r(x)$. Let us choose a maximal finite sequence x_1, \dots, x_p with this property.

Lemma 4.6. *The balls $B(x_1, \frac{2}{3}r(x_1)), \dots, B(x_p, \frac{2}{3}r(x_p))$ cover $M_n \setminus \text{int}(W_n)$.*

Proof. Let $x \in M_n \setminus \text{int}(W_n)$ be an arbitrary point. By maximality, there exists a point x_j such that $B(x, \frac{1}{4}r(x)) \cap B(x_j, \frac{1}{4}r(x_j)) \neq \emptyset$. From Lemma 4.5, we have $r(x) \leq \frac{4}{3}r(x_j)$ and $d(x, x_j) \leq \frac{1}{4}(r(x) + r(x_j)) \leq \frac{7}{12}r(x_j)$, hence $x \in B(x_j, \frac{2}{3}r(x_j))$. \square

Let us define $r_i := r(x_i)$. If $W_{n,1}, \dots, W_{n,q}$ are the components of W_n , so that the almost horospherical torus $\partial W_{n,i} \subset B(x_i, \frac{2}{3}r_i)$, we set:

- $V_i := B(x_i, r_i) \cup W_{n,i}$, for $i = 1, \dots, q$.
- $V_i := B(x_i, r_i)$, for $i > q$.

Furthermore, each component of $W_{n,i}$ can be retracted in order not to intersect V_j when $j \neq i$.

The construction of the open sets V_i and Lemma 4.6 imply the following:

Lemma 4.7. *The open sets V_1, \dots, V_p cover M_n .*

Let K be the nerve of the covering $\{V_i\}$. We will use this covering and the complex K to build the required map from M to a 2-dimensional complex. The idea is first to map M to K and then to improve this mapping by pushing it into the 2-skeleton $K^{(2)}$ of K .

The following Lemma shows that the dimension of K is bounded above by a uniform constant.

Lemma 4.8. *There exists a universal upper bound N on the number of open sets V_i which intersect a given V_k .*

Proof. If $V_i \cap V_k \neq \emptyset$, then $B(x_i, r_i) \cap B(x_k, r_k) \neq \emptyset$ and $B(x_i, r_i) \subset B(x_k, 2r_i + r_k) \subseteq B(x_k, 4r_k)$. On the other hand, for all $i_1 \neq i_2$ such that V_{i_1} and V_{i_2} intersect V_k one has $d(x_{i_1}, x_{i_2}) \geq \frac{1}{4}(r_{i_1} + r_{i_2}) \geq \frac{3}{8}r_k$. The number of V_i intersecting V_k is thus bounded above by:

$$\frac{\text{vol}(B(x_k, 4r_k))}{\text{vol}(B(x_i, \frac{3}{16}r_k))} \leq \frac{\text{vol}(B(x_i, 8r_k))}{\text{vol}(B(x_i, \frac{3}{16}r_k))} \leq \frac{\text{vol}(B(x_i, 11r_i))}{\text{vol}(B(x_i, \frac{r_i}{8}))}.$$

As $B(x_i, 11r_i)$ is included in a ball $B(x', r')$ with curvature $\geq -\frac{1}{(r')^2}$, by Bishop-Gromov inequality this is bounded above by:

$$\frac{v_{-\frac{1}{(r')^2}}(11r_i)}{v_{-\frac{1}{(r')^2}}(\frac{r_i}{8})} = \frac{v_{-1}(\frac{11r_i}{r'})}{v_{-1}(\frac{r_i}{8r'})} \leq N.$$

□

Let $\Delta^{p-1} \subset \mathbf{R}^p$ denote the standard unit simplex of dimension $p - 1$. With a partition of unity (ϕ_i) adapted to the (V_i) and with certain metric properties, we construct a map:

$$f = \frac{1}{\sum_i \phi_i}(\phi_1, \dots, \phi_p): M_n \rightarrow \Delta^{p-1} \subset \mathbf{R}^p,$$

We view K as a subcomplex of Δ^{p-1} , so that the range of f is contained in K , whose dimension is at most N . Moreover, f maps

the components of W_n onto distinct vertices of the 0-skeleton of K . We first estimate the Lipschitz constant of the map $f : M_n \rightarrow K$, by choosing the ϕ_i 's.

Lemma 4.9. *There exists $L_N > 0$ such that the partition of unity can be chosen so that the restriction $f|_{V_k}$ is $\frac{L_N}{r_k}$ -Lipschitz.*

Proof. Let $\tau : [0, 1] \rightarrow [0, 1]$ be an auxiliary function with Lipschitz constant bounded by 4, which vanishes in a neighbourhood of 0 and verifies $\tau|_{[\frac{1}{3}, 1]} \equiv 1$. Let us define $\phi_k := \tau(\frac{1}{r_k}d(\partial V_k, \cdot))$ on V_k and let us extend it trivially on M_n . Then ϕ_k is $\frac{4}{r_k}$ -Lipschitz.

Let $x \in V_k$. The functions ϕ_i have Lipschitz constant $\leq \frac{4}{3} \cdot \frac{4}{r_k}$, and all ϕ_i vanish at x except at most $N + 1$ of them. Since the functions

$$(y_0, \dots, y_N) \mapsto \frac{y_k}{\sum_{i=0}^N y_i}$$

are Lipschitz on $\{y \in \mathbf{R}^{N+1} \mid y_0 \geq 0, \dots, y_N \geq 0 \text{ and } \sum_{i=0}^N y_i \geq 1\}$, and each $x \in M_n$ belongs to some V_k with $d(x, \partial V_k) \geq \frac{r_k}{3}$, the conclusion follows. \square

We shall now inductively deform f by homotopy into the 3-skeleton $K^{(3)}$, while keeping the local Lipschitz constant under control.

Lemma 4.10. *For all $d \geq 4$ and $L > 0$ there exists $L' = L'(d, L) > 0$ such that the following assertion holds true:*

Let $g : M_n \rightarrow K^{(d)}$ be a $\frac{L}{r_k}$ -Lipschitz map defined on V_k and such that the pull-back of the open star of the vertex $v_{V_k} \in K^{(0)}$ is contained in V_k . Then g is homotopic rel $K^{(d-1)}$ to a map $\tilde{g} : M_n \rightarrow K^{(d-1)}$ with the same properties as g , L being replaced by L' .

Proof. It suffices to find a constant $\theta = \theta(d, L) > 0$ such that each d -simplex $\sigma \subset K$ contains a point z whose distance to $\partial\sigma$ and to the image of g is $\geq \theta$. In order to push g into the $(d-1)$ -skeleton, we compose it on σ with the radial projection from z . This increases the Lipschitz constant by a multiplicative factor bounded above by a function of $\theta(d, L)$, and decrease the inverse image of the open stars of the vertices.

If θ does not satisfy the required property for some d -simplex σ , then $image(g) \cap int(\sigma)$ contains a set of cardinality at least $C(d) \cdot \frac{1}{\theta^d}$ of points whose pairwise distances are $\geq \theta$. Let $A \subset M_n$ be a set containing exactly one point of the inverse image of each of these points. Since g maps W_n into the 0-skeleton, $A \subset B(x_k, r_k)$. As g is $\frac{L}{r_k}$ -Lipschitz on V_k , the distance between any two distinct points in A

is bounded below by $\frac{1}{L}r_k\theta$. Hence the cardinal of A is bounded above by

$$\begin{aligned} \frac{\text{vol}(B(x_k, r_k))}{\text{vol}(B(y, \frac{r_k\theta}{2L}))} &\leq \frac{\text{vol}(B(y, 2r_k))}{\text{vol}(B(y, \frac{r_k\theta}{2L}))} \leq \frac{v_{-\frac{1}{(r')^2}}(2r_k)}{v_{-\frac{1}{(r')^2}}(\frac{r_k\theta}{2L})} = \\ &= \frac{v_{-1}(2\frac{r_k}{r'})}{v_{-1}(\frac{\theta}{2L}\frac{r_k}{r'})} \leq C \left(\frac{L}{\theta}\right)^3, \end{aligned}$$

where y is any point in A . In order to apply Bishop-Gromov, we used the fact that $B(x_k, 11r_k)$ is included in a ball of radius r' with curvature $\geq -1/(r')^2$. The inequality $C(d) \cdot \frac{1}{\theta^d} \leq C \cdot (\frac{L}{\theta})^3$ gives a positive lower bound $\theta_0(d, L)$ for θ . Consequently, any $\theta < \theta_0$ has the desired property. \square

Lemma 4.11. *There exists a constant C such that if D is large enough, then*

$$\text{vol}(B(x_i, r_i)) \leq C \frac{1}{D} r_i^3 \quad \text{for all } i.$$

Proof. We know that $\text{vol}(B(x_i, \nu_{x_i})) \leq \frac{1}{D} \nu_{x_i}^3$. Furthermore, $r_i \geq \frac{\nu_{x_i}}{11}$ and $B(x_i, r_i)$ is included in a ball $B(x', r')$ with curvature $\geq -\frac{1}{r'^2}$. As $r' \geq r_i$, the curvature on $B(x_i, r_i)$ is $\geq -\frac{1}{r_i^2}$. The Bishop-Gromov inequality gives:

$$\frac{\text{vol}(B(x_i, \frac{\nu_{x_i}}{11}))}{v_{-\frac{1}{r_i^2}}(\frac{\nu_{x_i}}{11})} \geq \frac{\text{vol}(B(x_i, r_i))}{v_{-\frac{1}{r_i^2}}(r_i)}.$$

Equivalently,

$$\text{vol}(B(x_i, \frac{\nu_{x_i}}{11})) \geq \frac{\text{vol}(B(x_i, r_i))}{v_{-1}(1)} v_{-1}(\frac{\nu_{x_i}}{11r_i}) \geq \text{vol}(B(x_i, r_i)) \frac{1}{C} \left(\frac{\nu_{x_i}}{r_i}\right)^3,$$

for some uniform $C > 0$. Hence

$$\text{vol}(B(x_i, r_i)) \leq C \left(\frac{r_i}{\nu_{x_i}}\right)^3 \text{vol}(B(x_i, \frac{\nu_{x_i}}{11})) \leq C r_i^3 \frac{1}{D}.$$

\square

Finally we push f into the 2-skeleton.

Lemma 4.12. *For a suitable choice of $D > 1$, there exists a map $f^{(2)}: M_n \rightarrow K^{(2)}$ such that:*

i. $f^{(2)}$ is homotopic to f rel $K^{(2)}$.

- ii. The inverse image of the open star of each vertex $v_{V_k} \in K^{(0)}$ is contained in V_k .

Proof. The inverse image by f of the open star of the vertex $v_{V_k} \in K^{(0)}$ is contained in V_k . Using Lemma 4.10 several times, we find a map $f^{(3)} : M_n \rightarrow K^{(3)}$ homotopic to f and a universal constant \hat{L} such that $(f^{(3)})^{-1}(\text{star}(v_{V_k})) \subset V_k$ and $f|_{V_k}^{(3)}$ is $\frac{\hat{L}}{r_k}$ -Lipschitz.

It now suffices to show that no 3-simplex $\sigma \subset K$ can lie entirely in the image of $f^{(3)}$. Indeed, once we know this, we can push $f^{(3)}$ into the 2-skeleton of K using a central projection in each simplex, with centre in the complement of this image. Note that here no metric estimate is required in the conclusion.

Let us thus assume that there exists a 3-simplex σ contained in the image of $f^{(3)}$. The inverse image of $\text{int}(\sigma)$ by $f^{(3)}$ is a subset of the intersection of the V_j 's such that v_{V_j} is a vertex of σ . Let V_k be one of them. As $\text{vol}(f^{(3)}(V_k)) \leq \text{vol}(f^{(3)}(B(x_k, r_k)))$, Lemma 4.11 yields:

$$\text{vol}(\text{image}(f^{(3)}) \cap \sigma) \leq \text{vol}(f^{(3)}(V_k)) \leq \left(\frac{\hat{L}}{r_k}\right)^3 \text{vol}(B(x_k, r_k)) \leq C \hat{L}^3 \frac{1}{D}$$

with uniform constants C and \hat{L} . Hence, if D is sufficiently large, then one has $\text{vol}(\text{image}(f^{(3)}) \cap \sigma) < \text{vol}(\sigma)$. \square

The retraction used to push f into the 2-skeleton does not involve the 0-skeleton of K . As a consequence, the inverse images of the open stars of the vertices v_k still satisfy $(f^{(2)})^{-1}(\text{star}(v_{V_k})) \subset V_k$, and thus $(f^{(2)})^{-1}(\text{star}(v_{V_k}))$ is homotopically trivial in M . This proves the assertion and ends the proof by contradiction of Proposition 4.1. \square

4.3 End of the proof of Theorem 0.1

Proposition 4.1 implies:

Corollary 4.13. *There exists $D_0 > 0$ such that if $D > D_0$ and $n \geq n_0(D)$, then there exists a compact submanifold $\mathcal{V}_0 \subset M_n$ with the following properties:*

- (i) \mathcal{V}_0 is either a connected component of W_n or a tubular neighbourhood of the soul of the local model of some point $x_0 \in M_n \setminus \text{int}(W_n)$.
- (ii) \mathcal{V}_0 is a solid torus, a thickened torus or the twisted I -bundle on the Klein bottle.
- (iii) \mathcal{V}_0 is homotopically non-trivial in M_n .

Proof. We recall that each component of W_n is a solid torus. If one of them is homotopically non-trivial, then we choose it. Otherwise, by Proposition 4.1, there exists a point $x_0 \in M_n \setminus \text{int}(W_n)$ such that $B(x_0, \nu_{x_0})$ is homotopically non-trivial; one of the remarks following Proposition 3.1 shows that $B(x_0, \nu_{x_0})$ is necessarily a solid torus, a thickened torus or a twisted I -bundle over the Klein bottle. Indeed, S_0 can be neither a point nor a 2-sphere, otherwise $B(x_0, \nu_{x_0})$ would be homeomorphic to B^3 or $S^2 \times I$, which have trivial fundamental group. \square

As \mathcal{V}_0 is not contained in any 3-ball, each component Y of its complement is irreducible, hence a *Haken manifold* whose boundary is a union of tori. In particular, Y admits a geometric decomposition. Here is an important consequence of Thurston's hyperbolic Dehn filling theorem (cf. [5, Prop. 10.17], [6, Prop. 9.36]):

Proposition 4.14. *Let Y be a Haken 3-manifold whose boundary is a union of tori. Assume that any manifold obtained from Y by Dehn filling has vanishing simplicial volume. Then Y is a graph manifold.*

In order to prove that M_n is a graph manifold, it is sufficient to show that each component of $M_n \setminus \mathcal{V}_0$ is a graph manifold. To conclude the proof of Theorem 0.1, it suffices to show the following proposition:

Proposition 4.15. *For n large enough, one can find a submanifold \mathcal{V}_0 as above such that every Dehn filling on each component Y of $M_n \setminus \text{int}(\mathcal{V}_0)$ has vanishing simplicial volume.*

We choose the set \mathcal{V}_0 as follows:

- If some component of W_n is homotopically non-trivial, then we choose it as \mathcal{V}_0 .
- If all components of W_n are homotopically trivial, then there exists a point $x \in M_n \setminus \text{int}(W_n)$ such that $B(x, \nu_x)$ is homotopically non-trivial. We choose $x_0 \in M_n \setminus \text{int}(W_n)$ such that

$$\nu_0 = \nu_{x_0} \geq \frac{1}{2} \sup\{\nu(x) \mid \pi_1(B(x, \nu(x))) \rightarrow \pi_1(M_n) \text{ is non-trivial}\}.$$

Let S_0 be the soul of the local model $B(x_0, \nu_0)$. We choose \mathcal{V}_0 to be the metric open δ -neighbourhood with $0 < \delta < \frac{\nu_0}{D}$. After possibly shrinking W_n , one has $\mathcal{V}_0 \cap W_n = \emptyset$, as $\nu_0 \leq 1$.

We say that a subset $U \subset M_n$ is *virtually abelian relatively to \mathcal{V}_0* if the image in $\pi_1(M_n \setminus \mathcal{V}_0)$ of the fundamental group of each connected component of $U \cap (M_n \setminus \mathcal{V}_0)$ is virtually abelian.

We set:

$$\text{ab}(x) = \sup \left\{ r \left| \begin{array}{l} B(x, r) \text{ is virtually abelian relatively to } \mathcal{V}_0 \text{ and} \\ B(x, r) \text{ is contained in a ball } B(x', r') \text{ with} \\ \text{curvature} \geq -\frac{1}{(r')^2} \text{ and } \frac{\text{vol}(B(x', r'))}{(r')^3} \leq 1/D \end{array} \right. \right\}$$

and

$$r(x) = \min\left\{\frac{1}{11} \text{ab}(x), 1\right\}.$$

We are now led to prove the following assertion:

Assertion 4.16. *With this choice of \mathcal{V}_0 , for n large enough, M_n can be covered by a finite collection of open sets U_i such that:*

- *Each U_i is either contained in a component of W_n or in a ball $B(x_i, r(x_i))$ for some $x_i \in M_n^-(\varepsilon_n)$. In particular, U_i is virtually abelian relatively to \mathcal{V}_0 .*
- *The dimension of this covering is not greater than 2, and it is zero on \mathcal{V}_0 .*

Let us first show why this assertion implies Proposition 4.15.

Proof. The covering described in the assertion induces naturally a covering on every closed and orientable manifold \hat{Y} , obtained by gluing solid tori to ∂Y . It is a 2-dimensional covering by open sets which are virtually abelian and thus amenable in \hat{Y} . Gromov's vanishing theorem [17, §3.1], see also [24], then implies that the simplicial volume of \hat{Y} vanishes, which proves Proposition 4.15. \square

We now prove Assertion 4.16. The argument for the construction of a 2-dimensional covering by abelian open sets is similar to the one used in the proof of Proposition 4.1, replacing everywhere the triviality radius triv by the abelianity radius ab . There are, however, a few differences, which we now point out.

If \mathcal{V}_0 is a connected component of W_n , then for n large enough we choose points $x_0, x_1, \dots, x_q \in \partial W_n$, with $x_0 \in \partial \mathcal{V}_0$ in such a way that

- Every boundary component of W_n contains exactly one of the x_j 's.
- The balls $B(x_j, 1)$ are pairwise disjoint.
- Every $B(x_j, 1)$ has normalised volume $\leq \frac{1}{D}$ and sectional curvature close to -1 .
- Every $B(x_j, 1)$ is contained in a thickened torus (which implies that this ball is abelian).

Furthermore, going sufficiently far in the cusp and taking n large enough, one can assume that $B(x_j, \frac{1}{9}r(x_j))$ contains an almost horospherical torus corresponding to a boundary component of W_n . In this case the proof previously done applies without any change, since the dimension of the original covering and all those obtained by shrinking is zero on W_n (or on a set obtained by shrinking W_n).

From now on we shall assume that *all connected components of W_n are homotopically trivial*. We then choose $x_0 \in \mathcal{V}_0 \subset M_n \setminus \text{int}(W_n)$ as above, and points $x_1, \dots, x_q \in \partial W_n$ as before.

We complete the sequence x_0, x_1, \dots, x_q to a maximal finite sequence $x_0, x_1, x_2, \dots, x_p$ such that the balls $B(x_i, \frac{1}{4}r(x_i))$ are disjoint.

We set $r_i = r(x_i)$, and, if $W_{n,1}, \dots, W_{n,q}$ are the connected components of W_n , then we set

- $V_0 := B(x_0, r_0)$.
- $V_i := B(x_i, r_i) \cup W_{n,i}$, for $i = 1, \dots, q$.
- $V_i := B(x_i, r_i) \setminus \mathcal{V}_0$ for $i = q+1, \dots, p$.

After possibly shrinking W_n , we have $\mathcal{V}_0 \cap B(x_i, r_i) = \emptyset$ for $i = 1, \dots, q$, since $r_i \leq 1$ and $\mathcal{V}_0 \cap W_n = \emptyset$. It follows that $\mathcal{V}_0 \cap V_i = \emptyset$ for $i \neq 0$.

Under the hypothesis that the W_n are homotopically trivial, the following two lemmas deal with the difference with the previous proofs.

Lemma 4.17. *Each $x \in M_n$ belongs to some V_k such that $d(x, \partial V_k) \geq \frac{1}{3}r_k$.*

This lemma is used in the control of the Lipschitz constant of the characteristic map, Lemma 4.9. In order to prove it, we begin with the following remark:

Remark. If n is large enough, we have $\mathcal{V}_0 \subset B(x_0, \frac{r_0}{9})$.

The bounds on the diameter of S_0 , the distance to the base point, and the radius of the neighbourhood, give $\mathcal{V}_0 \subset B(x_0, \frac{3\nu_{x_0}}{D})$. Then the remark follows.

Proof of Lemma 4.17. If $x \in W_n$, then there is nothing to prove. We thus assume that $x \in M_n \setminus W_n$. If $x \in B(x_0, \frac{2}{3}r_0)$ we may choose $k = 0$. Let us then assume that $x \notin B(x_0, \frac{2}{3}r_0)$. There exists k such that $x \in B(x_k, \frac{2}{3}r_k)$. If V_k and V_0 are disjoint, then we are done. Hence we assume that $V_k \cap V_0 \neq \emptyset$. By the previous remark, one has:

$$d(x, \mathcal{V}_0) \geq d(x, x_0) - \frac{1}{9}r_0 \geq \frac{2}{3}r_0 - \frac{1}{9}r_0 \geq \frac{3}{4} \cdot \frac{5}{9}r_k > \frac{1}{3}r_k.$$

This implies that $d(x, \partial V_k) \geq \frac{1}{3}r_k$. □

The second difference is that the triviality radius satisfies $\text{triv}(x_i) \geq \nu_{x_i}$ by construction. Here we shall prove that $\text{ab}(x_i) \geq c\nu_{x_i}$ for a uniform $c > 0$. This is used in the proof of Lemma 4.11, where the inequality $\text{vol}(B_i) \leq C\frac{1}{D}r_i^3$ will still be true, but with a different constant C .

Lemma 4.18. *There exists $c > 0$ such that $r_i \geq c\nu_{x_i}$ for all i .*

Proof. One has $r_0 \geq \frac{1}{11}\nu_{x_0}$ by construction. For all $i > 0$, if $B(x_i, \frac{\nu_{x_i}}{11}) \cap \mathcal{V}_0 = \emptyset$, then $r_i \geq \frac{1}{11}\nu_{x_i}$. Hence we assume $B(x_i, \frac{\nu_{x_i}}{11}) \cap \mathcal{V}_0 \neq \emptyset$, and we claim that $d(x_i, \mathcal{V}_0) > c'\nu_{x_i}$ for a uniform $c' > 0$. Since $\mathcal{V}_0 \subset B(x_0, \frac{1}{9}r_0)$:

$$d(x_i, \mathcal{V}_0) \geq d(x_i, x_0) - \frac{1}{9}r_0 \geq \frac{1}{4}r_0 - \frac{1}{9}r_0 > \frac{1}{8}r_0 \geq \frac{1}{100}\nu_{x_0}. \quad (1)$$

We distinguish two cases, according to whether \mathcal{V}_0 is contained in $B(x_i, \nu_{x_i})$ or not.

If $\mathcal{V}_0 \subset B(x_i, \nu_{x_i})$, the image of $\pi_1(B(x_i, \nu_{x_i})) \rightarrow \pi_1(M_n)$ cannot be trivial, since the image of $\pi_1(\mathcal{V}_0) \rightarrow \pi_1(M_n)$ is not. In addition,

$$\nu_{x_i} \leq 2\nu_{x_0}, \quad (2)$$

by the choice of ν_{x_0} . Equations (1) and (2) give $d(x_i, \mathcal{V}_0) > \nu_{x_i}/200$.

If $\mathcal{V}_0 \not\subset B(x_i, \nu_{x_i})$, then since $\mathcal{V}_0 \cap B(x_i, \frac{\nu_{x_i}}{11}) \neq \emptyset$, we have

$$\text{diam}(\mathcal{V}_0) + \frac{\nu_{x_i}}{11} \geq \nu_{x_i}.$$

Consequently, $\text{diam}(\mathcal{V}_0)/\nu_{x_i} \geq 10/11$. As $\text{diam}(\mathcal{V}_0) \leq 3\nu_{x_0}/D$, we deduce that

$$\frac{\nu_{x_0}}{\nu_{x_i}} \geq \frac{10}{33}D.$$

Combining with (1), we get $d(x_i, \mathcal{V}_0) \geq \frac{D}{330}\nu_{x_i}$.

For $D > 30$, this contradicts $\mathcal{V}_0 \cap B(x_i, \frac{\nu_{x_i}}{11}) \neq \emptyset$. \square

5 Ricci flow with surgery

In order to apply Theorem 0.1, we shall need the following straightforward consequence of Perelman's work [31, 33, 32]:

Theorem 5.1. *Let M be a closed, orientable, irreducible 3-manifold.*

- (1) *If $\pi_1 M$ is finite, then M is spherical.*
- (2) *If $\pi_1 M$ is infinite, then for every riemannian metric g_0 on M , there exists an infinite sequence of riemannian metrics g_1, \dots, g_n, \dots with the following properties:*
 - (i) *The sequence $(\hat{R}(g_n))_{n \geq 0}$ is nondecreasing. In particular, it has a limit, which is greater than or equal to $\hat{R}(g_0)$.*
 - (ii) *The sequence $(\text{vol}(g_n))_{n \geq 0}$ is bounded.*
 - (iii) *Let $\varepsilon > 0$ be a real number and $x_n \in M$ be a sequence such that for all n , x_n is ε -thick with respect to g_n . Then the sequence (M, g_n, x_n) subconverges in the \mathcal{C}^2 topology towards some hyperbolic pointed manifold.*
 - (iv) *The sequence g_n has controlled curvature in the sense of Perelman.*

In this section we explain how to deduce Theorem 5.1 from Perelman's results on the Ricci flow. We refer to [27] and [29] for the details.

Let M be a closed, orientable 3-manifold. R. Hamilton introduced in [19] the following equation:

$$\frac{dg}{dt} = -2 \text{Ric}(g),$$

where the unknown $g = g(t)$ is a family of riemannian metrics on M depending on a time parameter $t \in \mathbf{R}$. A *Ricci flow* is a solution to this equation.

In [33], Perelman constructs an object he calls *Ricci flow with δ -cutoff*, also known as *Ricci flow with surgery*. It can be viewed as a 1-parameter family of (possibly disconnected) riemannian manifolds $(M(t), g(t))$ which satisfies Hamilton's equation in a weak sense. The topology of the manifold $M(t)$ is allowed to change at a discrete set of times, the change being a connected sum decomposition into prime factors, as well as RP^3 or $S^2 \times S^1$ factors, and removing components that are spherical or diffeomorphic to $S^2 \times S^1$.

Perelman [33, §1–5] (see also [27, §58–80], [29, Chapters 14–17]) shows that for every riemannian metric g_0 on M , there exists a Ricci flow with surgery satisfying the initial condition $(M(0), g(0)) = (M, g_0)$. It may happen that $M(t)$ becomes empty for some finite time t ; in

this case, M is a connected sum of spherical manifolds and copies of $S^2 \times S^1$. Such a Ricci flow with surgery is said to become *extinct*.

From now on we assume that M is irreducible. Thus, if some Ricci flow with surgery with initial manifold $M(0) = M$ becomes extinct, then M is spherical. If M is not diffeomorphic to S^3 and $(M(t), g(t))$ is a Ricci flow with surgery such that $M(0) = M$ and which does not become extinct, then for each time t , the manifold $M(t)$ has exactly one component diffeomorphic to M , the others being copies of S^3 . Thus we get a 1-parameter family of metrics on M , which we still denote by $g(t)$, defined for all $t \geq 0$. (There is some freedom for the choice of diffeomorphisms between M and the various $M(t)$'s, but the following discussion does not depend on this choice.)

If the metric g_0 has positive scalar curvature, then a maximal principle argument shows that any Ricci flow with surgery with initial metric g_0 becomes extinct in finite time. Thus M is spherical. The same conclusion holds if $\pi_1 M$ is finite (see [32], [29, Chapter 18], [12].)

If $\pi_1 M$ is infinite, then Ricci flow with surgery cannot become extinct, and M cannot admit any metric of positive scalar curvature. Let g_0 be any riemannian metric on M . By scaling, we get a *normalised* metric \hat{g}_0 (i.e. the absolute value of its sectional curvature is bounded above by 1, and each ball of radius 1 has volume greater than or equal to half of the volume of the Euclidean unit ball.) Starting with the initial metric \hat{g}_0 , we get a family of metrics $\{g(t)\}_{t \geq 0}$. For all integers $n \geq 1$, set $g_n := (4n)^{-1}g(n)$. Then it is proved in [33, §6–7] (see [27, §81–89] for details) that the sequence $\{g_n\}_{n \geq 0}$ satisfies Properties (ii)–(iv) of the conclusion of Theorem 5.1.³ Moreover, the function $t \mapsto \hat{R}(g(t))$ is nondecreasing [33, §7.1]. Since \hat{R} is scale invariant, we have $\hat{R}(g_0) = \hat{R}(\hat{g}_0)$, and $\hat{R}(g_n) = \hat{R}(g(n))$ for all $n \geq 1$. Hence the sequence $(\hat{R}(g_n))_{n \geq 0}$ is nondecreasing. This completes the proof of Theorem 5.1.

6 Applications

6.1 A sufficient condition for hyperbolicity

The following proposition is Assertion (1) of Theorem 0.3:

Proposition 6.1. *Let M be a closed, orientable and irreducible 3-manifold. Suppose that the inequality $\overline{R}(M) \leq -6V_0(M)^{2/3}$ holds. Then M is hyperbolic and the hyperbolic metric realises $\overline{R}(M)$.*

³ An immaterial difference between our statement and Perelman's is that we use the rescaling factor $(4t)^{-1}$ instead of t^{-1} in order to get limits of sectional curvature -1 rather than $-1/4$.

Proof. Let H_0 be a hyperbolic manifold homeomorphic to the complement of a link L_0 in M and whose volume realises $V_0(M)$. To prove Proposition 6.1, it is sufficient to show that L_0 is empty. Let us assume that it is not true and prove that M carries a metric g_ε such that $\text{vol}(g_\varepsilon) < V_0(M)$ and $R_{\min}(g_\varepsilon) \geq -6$. This can be done by a direct construction as in [2]. We give here a different argument relying on Thurston's hyperbolic Dehn filling theorem.

If $L_0 \neq \emptyset$, then we consider the orbifold \mathcal{O} with underlying space M , singular locus L_0 local group $\mathbb{Z}/n\mathbb{Z}$ with $n > 1$ sufficiently large so that the orbifold carries a hyperbolic structure, by the hyperbolic Dehn filling theorem [38] (cf. [7, Appendix B]). We then desingularise the conical metric on M corresponding to the orbifold structure, in a tubular neighbourhood of L_0 :

Lemma 6.2 (Salgueiro [35]). *For each $\varepsilon > 0$ there exists a Riemannian metric g_ε on M with sectional curvature bounded below by -1 and such that $\text{vol}(M, g_\varepsilon) < (1 + \varepsilon)^{\frac{5}{2}} \text{vol}(\mathcal{O})$.*

For completeness we give the proof of this lemma, following [35, Chap. 3].

Proof. Let g be the hyperbolic cone metric on M induced by the hyperbolic orbifold \mathcal{O} . Let $\mathcal{N} \subset \mathcal{O}$ be a tubular neighbourhood of radius $r_0 > 0$ around the singular locus L_0 . In \mathcal{N} the local expression of the singular metric g in Fermi (cylindrical) coordinates is:

$$ds^2 = dr^2 + \left(\frac{1}{n} \sinh(r) \right)^2 d\theta^2 + \cosh^2(r) dh^2,$$

where $r \in (0, r_0)$ is the distance to L_0 , h is the length parameter along L_0 , and $\theta \in (0, 2\pi)$ is the rescaled angle parameter.

The deformation depends only on the parameter r and consists in replacing the metric g by a smooth metric g' which coincides with g outside of \mathcal{N} , and has in \mathcal{N} the form

$$ds^2 = dr^2 + \phi^2(r) d\theta^2 + \psi(r)^2 dh^2,$$

where for some $\delta = \delta(\varepsilon) > 0$ sufficiently small the functions

$$\phi, \psi : [0, r_0 - \delta] \rightarrow [0, +\infty)$$

are smooth and satisfy the following properties:

- (1) In a neighbourhood of 0, $\phi(r) = r$ and $\psi(r)$ is constant.
- (2) In a neighbourhood of $r_0 - \delta$, $\phi(r) = \frac{1}{n} \sinh(r + \delta)$ and $\psi(r) = \cosh(r + \delta)$.

$$(3) \quad \forall r \in (0, r_0 - \delta), \frac{\phi''(r)}{\phi(r)} \leq 1 + \varepsilon, \quad \frac{\psi''(r)}{\psi(r)} \leq 1 + \varepsilon \text{ and } \frac{\phi'(r)\psi'(r)}{\phi(r)\psi(r)} \leq 1 + \varepsilon.$$

The new metric is non-singular by (1), it matches the previous one away from \mathcal{N} by (2) and has sectional curvature $\geq -1 - \varepsilon$ by (3).

First we deal with the construction of ϕ . Let $r_1 = r_1(\delta) > 0$ be defined by $r_1 = \frac{1}{n} \sinh(r_1 + \delta)$. Notice that $r_1 \approx \frac{1}{n-1} \delta$. The function ϕ is a smooth modification in a neighbourhood of r_1 of the piecewise smooth function

$$r \mapsto \begin{cases} r & \text{on } [0, r_1] \\ \frac{1}{n} \sinh(r + \delta) & \text{on } [r_1, r_0 - \delta] \end{cases}$$

so that:

- On $[0, r_1]$, $\phi \geq \frac{r}{1+\varepsilon}$, $\phi' \leq 1$, $\phi'' \leq 0$.
- On $[r_1, r_0 - \delta]$, $\phi \geq \frac{1}{n} \sinh(r + \delta) \frac{1}{1+\varepsilon}$, $\phi' \leq \frac{1}{n} \cosh(r + \delta)$, $\phi'' \leq \frac{1}{n} \sinh(r + \delta)$.

We choose ψ satisfying:

- On $[0, r_1]$, ψ is constant.
- On $[r_1, r_0 - \delta]$, $\psi \geq \cosh(r + \delta)$, $\psi' \leq \sinh(r + \delta)$, $\psi'' \leq \cosh(r + \delta)(1 + \varepsilon)$.

Notice that $\psi'(r_1) = 0$ and $\psi'(r_0 - \delta) = \sinh(r_0)$, so for a given $\varepsilon > 0$, one has to choose δ sufficiently small to achieve the required bound on ψ'' .

As $\delta \rightarrow 0$, $\text{vol}(M, g') \rightarrow \text{vol}(\mathcal{O})$, since $\phi \rightarrow \frac{1}{n} \sinh(r)$ and $\psi \rightarrow \cosh^2(r)$. So given $\varepsilon > 0$, for a choice of δ sufficiently small, one obtains a smooth Riemannian metric g' on M with sectional curvature $\geq -1 - \varepsilon$ and volume $\text{vol}(M, g') \leq (1 + \varepsilon) \text{vol}(\mathcal{O})$. Then the rescaled metric $g_\varepsilon = \sqrt{1 + \varepsilon} g'$ on M has sectional curvature ≥ -1 and volume $\text{vol}(M, g_\varepsilon) \leq (1 + \varepsilon)^{\frac{5}{2}} \text{vol}(\mathcal{O})$. \square

As $\text{vol}(\mathcal{O}) < \text{vol}(H_0)$, for $\varepsilon > 0$ sufficiently small we obtain a Riemannian metric on M such that $\text{vol}(M, g_\varepsilon) < \text{vol}(H_0)$ and $R_{\min}(g_\varepsilon) \geq -6$. In particular

$$\hat{R}(g_\varepsilon) > -6 \text{vol}(H_0)^{2/3} = -6V_0(M)^{2/3}$$

which contradicts the hypothesis. The link L_0 is thus empty and we have $M = H_0$. \square

6.2 Proof of Theorem 0.3

Thanks to previous section, there just remains to show Assertion (2). If $\pi_1 M$ is finite, then by Theorem 5.1(1), M is spherical, hence a graph manifold. From now on we assume that $\pi_1 M$ is infinite. In particular, M is not simply connected.

By assumption, there exists a riemannian metric g_0 on M such that $-6V_0(M)^{2/3} < \hat{R}(g_0)$. Applying Theorem 5.1(2), we get a sequence of metrics $\{g_n\}$ satisfying properties (i)–(iv) of this Theorem. Note that properties (ii) and (iv) are respectively Hypotheses (1) and (3) of Theorem 0.1. Next we check Hypothesis (2), which is the content of the following lemma:

Lemma 6.3. *If H is any hyperbolic 3-manifold which appears as a pointed \mathcal{C}^2 -limit of the some subsequence of (M, g_n) , then $\text{vol}(H) < V_0(M)$.*

Proof. Looking for a contradiction, we assume $\text{vol}(H) \geq V_0(M)$.

By monotonicity of $\hat{R}(g_n)$ and choice of g_0 , we have

$$\lim \hat{R}(g_n) \geq \hat{R}(g_0) > -6V_0(M)^{2/3}.$$

Let $\xi > 0$. We choose a compact core $\bar{H}(\xi)$ of H such that $\text{vol}(\bar{H}(\xi)) \geq (1 - \xi) \text{vol}(H)$. Since the convergence is \mathcal{C}^2 , there exists, for n sufficiently large, a submanifold $\bar{H}_n(\xi) \subset M_n$ with volume at least $(1 - \xi)^2 \text{vol}(H)$ and whose scalar curvature is less than or equal to $-6(1 - \xi)$. Thus, for n large, $R_{\min}(g_n) \leq -6(1 - \xi)$ and $\text{vol}(M_n) \geq \text{vol}(\bar{H}_n(\xi)) \geq (1 - \xi)^2 \text{vol}(H)$. Letting $\xi \rightarrow 0$ we get

$$\lim \hat{R}(g_n) \leq -6 \text{vol}(H)^{2/3} \leq -6V_0(M)^{2/3},$$

which gives the desired contradiction. \square

Hence we can apply Theorem 0.1. It follows that M contains an incompressible torus or is a graph manifold. In the latter case, M contains an incompressible torus or is Seifert fibred. \square

6.3 Proof of Theorem 0.4

Let us recall that $V'_0(M)$ denotes the infimum of the volumes of hyperbolic manifolds which can be embedded in M with an incompressible cusp in M or as complement of a (possibly empty) link in M . Since hyperbolic volumes form a well-ordered set, this infimum is in fact a minimum, hence $V'_0(M) > 0$.

Let M be a graph manifold. After Cheeger-Gromov [11], one can construct Riemannian metrics on M with sectional curvature pinched

between -1 and 1 whose volume is arbitrarily small. Thus $\overline{R}(M) \geq 0$. Since $V'_0(M) > 0$, this proves the ‘only if’ part of the equivalence.

The ‘if’ part follows from Theorem 5.1 and the following variant of Theorem 0.1:

Theorem 6.4. *Let M be a closed, orientable, non-simply connected, irreducible 3-manifold. Let g_n be a sequence of Riemannian metrics satisfying:*

- (1) *The sequence $\text{vol}(g_n)$ is bounded.*
- (2) *For all $\varepsilon > 0$, if $x_n \in M$ is a sequence such that for all n , x_n is in the ε -thick part of (M, g_n) , then (M, g_n, x_n) subconverges in the \mathcal{C}^2 topology to a pointed hyperbolic manifold with volume strictly less than $V'_0(M)$.*
- (3) *The sequence (M, g_n) has locally controlled curvature in the sense of Perelman.*

Then M is a graph manifold.

Proof. Let H^1, \dots, H^m be hyperbolic limits given by Proposition 2.1 and let $\varepsilon_n \rightarrow 0$ be a sequence chosen as in the remark after Proposition 2.1, or in the beginning of Section 3 to describe the local structure of the thin part. As in the proof of Theorem 6.4, for each i we fix a compact core \bar{H}^i of H^i and for each n a submanifold \bar{H}_n^i and an approximation $\phi_n^i : \bar{H}_n^i \rightarrow \bar{H}^i$. The fact that the volume of each hyperbolic manifold H^i is less than $V'_0(M)$ implies the following result:

Lemma 6.5. *Up to taking a subsequence of M_n , for all $i \in 1, \dots, m$ each component of $\partial \bar{H}_n^i$ is compressible in M for all n .*

Proof. Indeed if the conclusion of Lemma 6.5 does not hold, then up to extracting a subsequence, we may assume that there exists an integer $i_0 \in 1, \dots, m$ such that $\partial \bar{H}_n^{i_0}$ contains an incompressible torus for all n . From the definition of $V'_0(M)$ this would contradict the inequality $\text{vol}(H^{i_0}) < V'_0(M)$. \square

From this lemma on, the proof of Theorem 6.4 is identical to the proof of Theorem 0.1. \square

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Institut Fourier, Université de Grenoble I, UMR 5582 CNRS-UJF, 38402, Saint-Martin-d’Hères, France
 laurent.bessieres@ujf-grenoble.fr, g.besson@fourier.ujf-grenoble.fr

Institut Mathématique de Toulouse, UMR 5219 CNRS-UPS, Université Paul Sabatier, 31062 Toulouse Cedex 9, France.
 boileau@picard.ups-tlse.fr

Institut de Recherche Mathématique Avancée, Université Louis Pasteur, 7 rue René Descartes, 67084 Strasbourg Cedex, France
 maillot@math.u-strasbg.fr

Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain
 porti@mat.uab.es